DETERMINATION OF JOULE DISSIPATION FOR A LIQUID WITH VARIABLE CONDUCTIVITY MOVING IN AN INHOMOGENEOUS MAGNETIC FIELD

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There is a very wide class of conductors in which the conductivity cannot be regarded as independent of the current density. Obvious examples are plasmas and semiconductors. The physics of this phenomenon was investigated in [1, 2], where it was found that the conductivity is a function of the modulus of the current density. The dependence of the current on the electric field strength in Ohm's law becomes nonlinear. As a result, the equations of electrodynamics become nonlinear: they may be either elliptic or hyperbolic, depending on the type of dependence of the conductivity on the current. It was shown in [3] that relationships $\sigma = \sigma(j)$ which lead to the hyperbolic case have no physical significance. The equations are elliptic when the function $\sigma = \sigma(j)$ satisfies the condition

$$\sigma - j\frac{d\sigma}{dj} > 0 .$$

In many practical cases the conductivity is only weakly dependent on the current; thus

$$\sigma = \sigma_0 + \varepsilon\sigma_1(j), \quad \sigma_0 = \text{const} \tag{0.1}$$

where $\sigma_1(j)$ is a differentiable function and ε is a small parameter. The small-parameter method may then be used for finding the unknowns. This method is used (e. g., [4]) for finding, in first approximation in ε, the Joule loss in the region close to an electrode with a constant magnetic field. In addition to the Joule loss, losses due to edge effects are also possible. In the channels of an MHD generator, conditions may be realized in which the magnetic field remains uniform for a considerable distance from the electrode region, beyond which it falls rapidly to zero. The closed currents that lead to the extra losses arise at the points where the electrically conducting medium enters and leaves the magnetic field. If the length of the uniform-field section outside the electrode zone is more than twice the channel width, we can obtain a good approximation to the problem of end effects where the medium enters and leaves the generator channel by considering the current distribution in a conducting medium moving in an infinitely long channel with dielectric walls in the presence of a magnetic field which is constant in half of the channel and zero in the other half. The Joule loss in a channel with parallel walls was calculated in [5] for the case of constant conductivity.

The same problem will be considered below, except that the conductivity will be assumed weakly dependent on the current. An expression is obtained for the Joule loss to first approximation in ε; this expression only contains the zero approximation for the current. The equations of electrodynamics are always elliptic when the conductivity depends weakly on the current. We also perform a numerical computation from our expression for a concrete function of the (0.1) type, namely

$$\sigma_1(j) = \sigma_1 j^2 / j_*^2, \quad \sigma_1 = \text{const}.$$

Here j_* denotes some characteristic current.

1. Consider a two-dimensional steady-state flow of conducting liquid in a channel with insulated walls (Fig. 1). A constant magnetic field $B = (0, 0, B_0)$ with vector directed towards the reader is applied in the right-hand side of the channel. The magnetic field in the left side of the channel is zero. Combining these two conditions, we get

$$\mathbf{B} = (0,\ 0,\ B), \quad B = B_0\theta(x), \quad B_0 = \text{const},$$
$$\theta(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x < 0). \end{cases} \tag{1.1}$$

The liquid flows in the channel with constant velocity

$$\mathbf{V} = (u_0,\ 0,\ 0), \quad u_0 = \text{const}. \tag{1.2}$$

We shall solve the problem in the approximation in which the hydrodynamic quantities can be assumed known.

Using Ohm's law and the fact that the electric field is potential, and introducing the stream function ψ with the current density j:

$$\mathbf{j} = (\sigma_0 + \varepsilon\sigma_1(j))\left(\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B}\right) \qquad \left(j_x = \frac{\partial\psi}{\partial y},\ j_y = -\frac{\partial\psi}{\partial x}\right), \qquad \mathbf{E} = -\operatorname{grad}\varphi \tag{1.3}$$

we get the equation for ψ:

$$\operatorname{div}\left[\frac{1}{\sigma_0 + \varepsilon\sigma_1(j)}\operatorname{grad}\psi\right] = -\frac{u_0 B_0 \delta(x)}{c}. \tag{1.4}$$

Here $\delta(x)$ is the Dirac delta function.

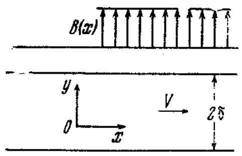

Fig. 1

On the insulated channel walls we have the boundary condition $j_n = 0$. No currents flow at infinity, i.e., $\mathbf{j}|_{x=\pm\infty} = 0$. For ψ we have

$$\left.\frac{\partial\psi}{\partial x}\right|_{y=\pm\delta} = 0, \qquad \left.\frac{\partial\psi}{\partial x}\right|_{x=\pm\infty} = 0, \qquad \left.\frac{\partial\psi}{\partial y}\right|_{x=\pm\infty} = 0. \tag{1.5}$$

We seek the solution of (1.4) under the boundary conditions (1.5) in the form

$$\psi = \sum_{k=0}^{\infty}\frac{1}{k!}\psi_k\varepsilon^k \qquad \left(\psi_k = \left.\frac{\partial^k\psi}{\partial\varepsilon^k}\right|_{\varepsilon=0},\ j_{xk} = \frac{\partial\psi_k}{\partial y},\ j_{yk} = -\frac{\partial\psi_k}{\partial x}\right).$$

Here, ψ_0 is the zero approximation for ψ, and $\psi_0 + \varepsilon\psi_1$ the first approximation. Putting $\varepsilon = 0$ in Eq. (1.4) and boundary conditions (1.5), we get the following equation and boundary conditions for ψ_0:

$$\Delta\psi_0 = -\frac{\sigma_0 u_0 B_0 \delta(x)}{c}, \tag{1.6}$$

$$\left.\frac{\partial\psi_0}{\partial x}\right|_{y=\pm\delta} = 0, \qquad \left.\frac{\partial\psi_0}{\partial x}\right|_{x=\pm\infty} = 0, \qquad \left.\frac{\partial\psi_0}{\partial y}\right|_{x=\pm\infty} = 0. \tag{1.7}$$

On differentiating (1.4) and boundary conditions (1.5) with respect to ε, then putting $\varepsilon = 0$, we get the following equation and boundary conditions for ψ_1:

$$\Delta\psi_1 = \operatorname{div}\left(\frac{\sigma_1(j_0)}{\sigma_0}\operatorname{grad}\psi_0\right), \tag{1.8}$$

$$\left.\frac{\partial\psi_1}{\partial x}\right|_{y=\pm\delta} = 0, \qquad \left.\frac{\partial\psi_1}{\partial x}\right|_{x=\pm\infty} = 0, \qquad \left.\frac{\partial\psi_1}{\partial y}\right|_{x=\pm\infty} = 0. \tag{1.9}$$

Solving (1.6) under boundary conditions (1.7), we obtain the zero approximation for the current density j_0:

$$j_{x0} = -\frac{\sigma_0 u_0 B_0}{2\pi c}\ln\frac{\operatorname{ch}\pi x/2\delta + \sin\pi y/2\delta}{\operatorname{ch}\pi x/2\delta - \sin\pi y/2\delta},$$

$$j_{y0} = -\frac{\sigma_0 u_0 B_0}{\pi c}\operatorname{arctg}\left[\operatorname{sh}\frac{\pi x}{2\delta}\sec\frac{\pi y}{2\delta}\right] + \frac{\sigma_0 u_0 B_0}{c}\left(\theta(x) - \frac{1}{2}\right). \tag{1.10}$$

Notice also that, as $|x| \to \infty$, the exact solution of (1.4) under boundary conditions (1.5) tends to a constant (i.e., the current density tends to zero), since the effect produced by nonuniformity of the magnetic field becomes negligible when $|x|$ is large.

We now turn to finding the Joule dissipation Q in the first approximation. The exact expression for Q is

$$Q = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta} \frac{j^2}{\sigma(j)}\, dydx\,.$$

In view of this, the dissipation is, in first approximation,

$$P = Q_0 + \varepsilon Q_1\,,$$

$$Q_0 = Q\,|_{\varepsilon=0}, \qquad Q_0 = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta} j_0^2/\sigma_0 dydx\,,$$

$$Q_1 = \frac{\partial Q}{\partial \varepsilon}\Big|_{\varepsilon=0}, \qquad Q_1 = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left[\frac{2}{\sigma_0}(\mathrm{j}_0,\ \mathrm{j}_1) - \frac{\sigma_1(j_0)}{\sigma_0^2} j_0^2\right] dydx\,. \tag{1.11}$$

Differentiating (1.3) with respect to ε and putting $\varepsilon = 0$, we get

$$\frac{1}{\sigma_0}\mathrm{j}_1 - \frac{\sigma_1(j_0)}{\sigma_0^2}\mathrm{j}_0 = -\operatorname{grad}\varphi_1\,,$$

and on using (1.8), we get

$$Q_1 = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left[\frac{2}{\sigma_0}(\operatorname{grad}\psi_0,\ \operatorname{grad}\psi_1) - \frac{\sigma_1(j_0)}{\sigma_0^2} j_0^2\right] dydx =$$

$$= \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left[\frac{2}{\sigma_0}\operatorname{div}(\psi_0 \operatorname{grad}\psi_1) - \frac{2}{\sigma_0}\psi_0\Delta\psi_1 - \frac{\sigma_1(j_0)}{\sigma_0^2} j_0^2\right] dydx =$$

$$= \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left[\frac{2}{\sigma_0}\operatorname{div}(\psi_0 \operatorname{grad}\psi_1) - \frac{2}{\sigma_0}\psi_0 \operatorname{div}\left(\frac{\sigma_1(j_0)}{\sigma_0}\operatorname{grad}\psi_0\right) - \frac{\sigma_1(j_0)}{\sigma_0^2} j_0^2\right] dydx =$$

$$= \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\operatorname{div}\left[2\psi_0\left(\frac{1}{\sigma_0}\operatorname{grad}\psi_1 - \frac{\sigma_1(j_0)}{\sigma_0^2}\operatorname{grad}\psi_0\right)\right] dydx + \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{\sigma_1(j_0)}{\sigma_0^2} j_0^2\, dydx =$$

$$= 2\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\operatorname{div}\left[-\frac{\partial(\psi_0\varphi_1)}{\partial y}\mathfrak{s}_x + \frac{\partial(\psi_0\varphi_1)}{\partial x}\mathfrak{s}_y\right] dydx +$$

$$+ \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\operatorname{div}(2\varphi_1\mathrm{j}_0)\, dydx + \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{\sigma_1(j_0)}{\sigma_0^2} j_0^2 dydx = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{\sigma_1(j_0)}{\sigma_0^2} j_0^2 dydx\,.$$

Here, $\mathfrak{s}_x$ and $\mathfrak{s}_y$ are the unit base vectors.

We can show that

$$2\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\operatorname{div}(\varphi_1\mathrm{j}_0)\, dydx = 0\,.$$

Using Green's theorem for the rectangle with vertices $A(-a, -\delta)$, $B(a, -\delta)$, $C(a, \delta)$, and $D(-a, \delta)$, together with (1.7) and the fact that $\mathrm{j}_0 \to 0$ as $x \to \pm\infty$, we have

$$2\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\operatorname{div}(\varphi_1\mathrm{j}_0)\, dydx = \lim_{a\to\infty} 2\int_{-a}^{a}\int_{-\delta}^{\delta}\operatorname{div}(\varphi_1\mathrm{j}_0)\, dydx =$$

$$= 2\lim_{a\to\infty}\left[\int_{AB}(-\varphi_1 j_{y0})\, dx + \int_{BC}\varphi_1 j_{x0} dy + \int_{CD}(-\varphi_1 j_{y0})\, dx + \int_{DA}\varphi_1 j_{x0} dy\right] = 0$$

since the integrals over AB and CD vanish due to the boundary condition $j_{y0}|_{y=\pm\delta} = 0$, while the limits of the integrals over BC and DA vanish as $a \to \infty$, since $\mathrm{j}_0 \to 0$ as $x \to \pm\infty$.

Hence the expression for the Joule dissipation in first approximation is

$$P = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{\sigma_0 + \varepsilon\sigma_1(j_0)}{\sigma_0^2} j_0^2 dydx = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{\sigma(j_0)}{\sigma_0^2} j_0^2 dydx\,. \tag{1.12}$$

To find P, we only need to know the zero approximation for the current density j_0.

Similar expressions may be obtained in the case of a two- or three-dimensional channel of arbitrary finite cross-section, with no restrictions on the velocity field or the magnetic field (the integration is performed over the region occupied by the liquid), provided that the sole condition is satisfied, that the currents vanish at infinity.

2. As an example, take the following particular case of (0.1):

$$\sigma = \sigma_0 + \varepsilon\sigma_1 j^2 / j_*^2.$$

Here, j_* is some characteristic current density,

$$\sigma_1 = \text{const}, \qquad P = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta} \frac{1}{\sigma_0} j_0^2 dy dx + \varepsilon \int_{-\infty}^{\infty}\int_{-\delta}^{\delta} \frac{\sigma_1}{\sigma_0^2 j_*^2} j_0^4 dy dx. \tag{2.1}$$

The first integral is evaluated in [5]:

$$Q_0 = \int_{-\infty}^{\infty}\int_{-\delta}^{\delta} \frac{1}{\sigma_0} j_0^2 dy dx = \frac{16\sigma_0\delta^2}{c^2\pi^3} u_0^2 B_0^2\, 1.052, \qquad Q_1 = \frac{\sigma_1}{\sigma_0^2 j_*^2} \int_{-\infty}^{\infty}\int_{-\delta}^{\delta} j_0^4 dy dx .$$

We perform the changes of variables

$$\tau = \frac{1}{2} \ln \frac{\operatorname{ch}(\pi x / 2\delta) + \sin(\pi y / 2\delta)}{\operatorname{ch}(\pi x / 2\delta) - \sin(\pi y / 2\delta)}, \qquad \omega = -\operatorname{arc\,tg}\left[\operatorname{sh}\frac{\pi x}{2\delta} \sec\frac{\pi y}{2\delta}\right], \tag{2.2}$$

and take the dimensionality factor outside the sign of the double integral; we then get

$$Q_1 = \frac{\sigma_1\sigma_0^2 4\delta^2 u_0^4 B_0^4}{j_*^2\pi^2 c^4} K, \qquad Q_1 = \frac{\sigma_1\sigma_0^2 4\delta^2 u_0^4 B_0^4}{j_*^2\pi^2 c^4} 3.019, \tag{2.3}$$

$$K = \frac{4}{\pi^2} \int_0^{\infty}\int_0^{1/2\pi} (\tau^2 + \omega^2)^2 \frac{d\omega d\tau}{\operatorname{ch}^2\tau - \cos^2\omega}, \tag{2.4}$$

$$K = \frac{93}{16\pi} \zeta(5 + \frac{5\pi^2 G}{12} - \frac{\pi^3}{24} 2.06355 \approx 3.019.$$

The final expression is

$$P = \frac{16\sigma_0\delta^2}{c^2\pi^3} u_0^2 B_0^2\, 1.052 + \varepsilon \frac{\sigma_1\sigma_0^2 4\delta^2 u_0^4 B_0^4}{j_*^2\pi^2 c^4} 3.019. \tag{2.5}$$

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